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MODELLING OF WAVE PROPAGATION IN THE NEARSHORE REGION USING THE MILD SLOPE EQUATION WITH GMRES-BASED ITERATIVE SOLVERS

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SUMMARY

The mild slope equation in its linear and non-linear forms is used for the modelling of nearshore wave propagation. The finite difference method is used to descretize the governing elliptic equations and the resulting system of equations is solved using GMRES-based iterative method. The original GMRES solution technique of Saad and Schultz is not directly applicable to the present case owing to the complex coefficient matrix. The simpler GMRES algorithm of Walker and Zhou is used as the core solver, making the upper Hessenberg factorization unneccessary when solving the least squares problem. Several preconditioning-based acceleration strategies are tested and the results show that the GMRES-based iteration scheme performs very well and leads to monotonic convergence for all the test-cases considered.

KEY WORDS: GMRES; mild slope equation; iterative solvers

1. INTRODUCTION

Berkhoff's mild slope equation[1] has resulted in significant progress in terms of wave prediction in the nearshore region. The original equation is applicable to linear waves propagating over variable sea bed geometry with 'mild' slopes, but weak non-linearity can also be empirically incorporated[2] and the equation has also been shown to give acceptable results for steep slopes.[3] However, the equation manifests itself in an elliptic form which can be realistically solved only over a limited area in plan owing to computational time and storage limitations. In order to deal with problems associated with large areas in plan, some simplifications are usually made which compromise the physics, such as the parabolic approach which imposes restrictions on the angle of wave propagation and in addition ignores wave reflections. Another solution strategy involves using a transient form of the mild slope equation, which leads to hyperbolic formulations in which a steady state solution is sought. Hyperbolic formulations require small Δt in order to satisfy constraints based on the Courant number. Such formulations are often characterized by a deterioration of the results with time due to numerical errors being introduced by the inexact boundary conditions.

Recent developments pertaining to the iterative solution of linear systems of equations enable efficient solution of elliptic problems similar to that typically described by the mild slope equation. Conjugate-gradient-based methods usually produce rapid convergence to the solution of a linear group of equations and are also economical in terms of computer storage because they generally require only a matrix–vector product to be kept in memory rather than the complete coefficient matrix. With regard to the mild slope equation this attribute enables solutions to be obtained for large areas in plan[4] without any trade-off in terms of the description of the physics of the problem in hand.

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In this paper an iterative solution of the mild slope equation in its linear form (a Helmholtz-type equation) is described. Also, a non-linear form of the same equation is solved which uses coarser discretizations, thus making it possible to deal with even larger areas in plan. The core elliptic solver is similar to the s-GMRES solution technique as proposed by Walker and Zhou.[5] The reason for using the s-GMRES method instead of the original GMRES of Saad and Schultz[6] is that in the s-GMRES scheme the upper Hessenberg factorization is no longer required, thus making it much simpler to deal with a matrix with complex coefficients. The adopted scheme is further extended using ideas from the FGMRES scheme[7] as well as the GMRESR scheme[8] and it also incorporates a range of preconditioners, including a simple preconditioner based on ILU factorization.

The paper is organized as follows. The governing equations are presented and discussed first. The mild equation is discussed in its original form and also in a transformed form as proposed by Li and Anastasiou.[9] This is followed by the presentation of the solution method based on the s-GMRES algorithm. Various preconditioning schemes, including ILU-GMRES, FGMRES and GMRESR, are also presented and discussed. Two test-cases are examined in Section 4. The first is Berkhoff's shoal case, for which results from other numerical models are available as well as experimental data. The second test-case is a harbour resonance problem, for which model results can be compared with an analytical solution. It is shown that the proposed solution method compares favourably with hitherto available numerical schemes.

2. GOVERNING EQUATIONS AND DISCRETIZATION

2.1. Governing Equations

The governing equation is the mild slope equation as proposed by Berkhoff,[1]

$$\nabla(cc_g\nabla\phi) + cc_g\phi = 0, \tag{1}$$

where c is the wave celerity, c_g is the wave group velocity and ϕ is the wave potential. This equation can be written as a Helmholtz equation[10]

$$\nabla^2\phi + k_c^2\phi = 0, \tag{2}$$

where

$$k_c^2 = k^2 - \frac{\nabla^2(cc_g)^{0.5}}{(cc_g)^{0.5}}, \tag{3}$$

with k the wave number.

The Helmholtz equation has the following characteristics.

1. The coefficients are complex, so the matrix derived from the numerical discretization is also complex.
2. The real parts of some eigenvalues are negative and, moreover, some of the eigenvalues are small. These characteristics generally lead to slow convergence of CG method.[11]
3. The matrix is non-Hermitian owing to the fact that the boundary conditions are usually of the Neumann type.
4. The finite difference discretization provides a poor resolution of the group velocity. Thus better resolution is required in order to achieve reliable results.

Using a transformation suggested by Radder,[10]

$$\phi = \exp(\psi), \tag{4}$$

the following non-linear equation can be derived, which was used as the modelling equation in the multigrid model of Li and Anastasiou:[9]

$$\nabla^2\psi + \nabla\psi \cdot \nabla\psi + k_c^2 = 0. \tag{5}$$

It should be noted that the above transformation is not valid at wave-crossing points, as pointed out by Radder,[12] where numerical instabilities might arise. A numerical dissipation mechanism, such as that caused by an upwind scheme, may be required at such points.

2.2. *Numerical Discretization*

The linear equation can be easily discretized using a central finite difference scheme as

$$\frac{\phi_{i,j+1} + 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2} + \frac{\phi_{i+1,j} + 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + k_c^2 = 0. \tag{6}$$

The discretized equations from a group of linear equations and applicable iterative solvers are described in Section 3.

The non-linear equation (5) is discretized as

$$\frac{\psi_{i,j+1} + 2\psi_{i,j} + \psi_{i,j-1}}{\Delta y^2} + \frac{\psi_{i+1,j} + 2\psi_{i,j} + \psi_{i-1,j}}{\Delta x^2} + F_y\frac{\delta_y/\psi_{i,j}}{\Delta y} + F_x\frac{\delta_x\psi_{i,j}}{\Delta x} + k_c^2 = 0, \tag{7}$$

where

$$F_x = \frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x}, \tag{8}$$

$$F_y = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y} \tag{9}$$

and δ is the upwind operator in the x- or y-direction, depending on the local wave direction calculated at the previous iteration step.

2.3. *Boundary Conditions*

The boundary conditions are first-order radiation boundary conditions. For the offshore driving boundary the following equation is used:

$$\frac{\partial\phi}{\partial x} = ik_c(2\phi_i - \phi), \tag{10}$$

where ϕ_i is the incident wave at the boundary.

At the downstream boundary the radiation boundary condition used reads

$$\frac{\partial\phi}{\partial x} = ik_c\phi. \tag{11}$$

In practice, such first-order boundary conditions often produce significant unwanted numerical reflections. In order to deal with this problem, either second-order schemes may be used[13] or some empirical coefficients can be introduced. The latter approach is used in the present scheme and the radiation boundary condition is simply written as

$$\frac{\partial\phi}{\partial x} = ik_c\alpha\phi, \tag{12}$$

where the coefficient α is obtained from a pure wave refraction calculation. The radiation boundary condition is discretized as

$$\frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} - ik_c\alpha\frac{\phi_{i+1,j} + \phi_{i,j}}{2} = 0. \tag{13}$$

3. ITERATIVE SOLVERS

The GMRES scheme proposed by Saad and Schultz[6] has become a very popular iterative method for the solution of linear systems of equations. The basic method is to evaluate a Krylov subspace using a modified Gram–Schmid process and the residue is minimized in this subspace. It is well established that full GMRES is optimal in terms of matrix–vector product count. GMRES is usually truncated to step m (denoted as GMRES(m)) because of (i) the quadratic increase in operation count of the modified Gram–Schmidt process and (ii) the loss of orthogonality due to the accumulation of round-off errors.

Many variants of the GMRES(m) method exist. The traditional simple preconditioning strategies such as ILU and SSOR are certainly applicable. However, more complicated strategies have hitherto been proposed. In particular, the preconditioner can be constructed in a different form for each iteration step, making use of available information on the current residue and the eigenvalue distribution given by the Arnoldi process at the previous iteration step. Such approaches produce variants of GMRES(m) such as FGMRES (flexible GMRES),[7] GMRESR (GMRES residue)[8] and ET-p-GMRES (eigenvalue-transferred preconditioning GMRES).[14] The preconditioner in the FGMRES technique can be a direct solver or even an iterative one, including GMRES itself, in order to accelerate convergence. The preconditioner in GMRESR uses a low-rank update of the original matrix in order to accelerate convergence. A typical realization of GEMRESR involves using the GMRES(m) scheme as a preconditioner to the CGR(k) scheme. The ET-p-GMRES scheme uses a series of low-rank transformations to transform the already approximated sparse eigenvalues so that they lie in the vicinity of point 1 in the complex plane. This last method seems the most promising in that the preconditioner is tailor-made for the problem and both the negative eigenvalues and small eigenvalues which are critical to the convergence of a CG-like method may be moved to more favourable positions in the complex plane. However, the number of required low-rank transformations may be large, the vectors of these low-rank transformations have to be stored for sparse matrices and this increases considerably both memory and multiplication count requirements as the number of transformations increases. Therefore we choose three typical preconditioning strategies, namely ILU, FGMRES and GMRESR, although other direct strategies or iterative ones such as the multigrid method can also be easily incorporated.

As discussed in the previous section, the linear system of equations which must be solved involves complex coefficients and the direct application of the GMRES method will not satisfy the orthogonality requirements. It should be noted, however, that the QR factorization is no longer necessary for one variant of GMRES, namely GMRES proposed by Walker and Zhou.[5] This scheme was originally devised to same some computational steps for a problem with a matrix of real coefficients and is adapted here to solve the complex matrix corresponding to the present problem, as explained in the following subsection.

3.1. The s-GMRES scheme

The s-GMRES scheme is formed by shifting the Arnoldi process to start with Ar_0 instead of starting with r_0 which is the original Krylov subspace construction suggested by Saad and Schultz.[6] The matrix thus formed is not an upper Hessengerg matrix but an upper triangular matrix. The elimination can be

directly applied to the upper triangular matrix in order to obtain the minimized vector. In the original GMRES scheme the QR factorization is usually realized via Givens rotations. However, for a matrix with complex coefficients the Givens rotations often do not lead to the minimization of the residue and they are not unitary. It is much simpler to use the s-GMRES scheme which does not require this QR factorization.

The s-GMRES scheme can be formulated as follows, noting that owing to the complex matrix some steps are different from those in the original algorithm proposed by Walker and Zhou.[5] Furthermore, the steps of the present algorithm are organized so as to enable incorporation of the FGMRES method as a preconditioner.

Algorithm.

1. Start $r_0 = b - Ax_0$, $\beta = \|r_0\|_2$, $r_0 = r_0/\beta$
2. Arnold process
   for $j = 1, \ldots, m$ do
   - $v_j = Av_{j-1} (v_1 = Ar_0)$
   - For $i = 1, \ldots, j-1$,
     $h_{i,j} = (v_j, v_i)$
     $v_j = v_j - h_{i,j} \ldots v\_i$
   - Compute $h_{j,j} = \|v_j\|_2$
     $v_j = v_j / h_{j,j}$
   - $p_j = (r, v_j)$
     $r = r - p_j \cdot v_j$
3. Solve $H_m y_m = P$
   where $P = (p_1, p_2, \ldots, p_m)^T$
   Form the approximate solution;
   $x_m = x_0 + V_m y_m$
4. Restart,
   if satisfied, stop else $x_0 = x_m$, goto step 1.

Here $\|v_j\|_2$ denotes the 2-norm of the vector v_j (see e.g. Reference 15); that is, if $v_j = (q_1, q_2, \ldots, q_n)^T$, then

$$\|v_j\|_2 = \sqrt{\left(\sum_{i=1}^{n} q_i\right)}. \tag{14}$$

The algorithm for a real coefficient matrix is discussed in detail by Walker and Zhou,[5] who have shown that it is equivalent to the general GMRES method with some additional small savings in terms of computational run-time due to the fact that the QR factorization is no longer required.

When applying the s-GMRES to the mild slope equation, all the coefficients in the matrix are scaled using the corresponding diagonal coefficient in order to eliminate possible small eigenvalues caused by the nature of the imposed boundary conditions.[11,16]

3.2. The Preconditioners

For the purposes of the problem at hand, three preconditioners are implemented, namely ILU, FGMRES (GMRES-GMRES) and GMRESR (GMRES-GCR).

3.2.1. The ILU Preconditioner. The incomplete $LD^{-1}U$ as proposed by Van de Vorst[17] is constructed in the following way.

1. Diag $(L) = \text{diag}(U) = D$.

2. The off-diagonal parts of L and U are equal to the corresponding parts of A.
3. Diag $(LD^{-1}U) = \text{diag}(A)$.

In this way, only the extra vector is required in order to produce the required preconditioner.

If the area of interest is rectangular, the preconditioner is even simpler, since the matrix A for the case of the mild slope equation is a pentadiagonal matrix with $a_{i,1}, \ldots, a_{i,5}$ being the non-zero element of row i. For this case the coefficient matrix has the property

$$d_i = a_{i,3} - a_{i-1,4}a_{i,2}d_{i-1}^{-1} - a_{i-m,5}a_{i,1}d_{i-m}^{-1}, \tag{15}$$

where m is the distance between $a_{i,5}$ and $a_{i,3}$ in the matrix A.

The ILU preconditioning approach uses the same matrix in every iteration but does not take advantage of the current residue available in the way both FGMRES and GMRESR do.

3.2.2. The FGMRES Preconditioner. The FGMRES scheme proposed by Saad[7] is a very flexible scheme from the point of view of incorporating preconditioners. It is possible to incoporate any iterative scheme in the solver as a preconditioner. The equation to solve is

$$AM^{-1}(Mx) = b. \tag{16}$$

As in any CG-like method, only the matrix–vector product is required. Whenever AM is required, $Mz = v$ is solved first and then the matrix A is applied to z. Instead of using the exact M as a traditional preconditioner requires, the approximate solution of $Mz = v$ can be used. This effectively takes advantage of the current residue when the new preconditioner is constructed, which leads to a more efficient algorithm. The algorithm in the present implementation of s-GMRES reads as follows

Algorithm

1. Start $r_0 = b - Ax_0$, $\beta = \|r_0\|_2$
2. Arnoldi process
   for $j = 1, \ldots, m$, do
   - $z_j = M^{-1}v_{j-1}$ $(z_1 = M^{-1}r_0)$
   - $w = Az_j$
   - For $i = 1, \ldots, j-1$,
     $h_{i,j} = (w, v_i)$
     $v_j = w - h_{i,j} \cdot v_i$
   - Compute $h_{j,j} = \|v_j\|_2$
     $v_j = w/h_{j,j}$
   - $p_j = (r, v_j)$
     $r = r - p_j \cdot v_j$
3. Solve $H_m y_m = P$
   where $P = (p_1, p_2, \ldots, p_m)^T$
   Form the approximate solution;
   $x_m = x_0 + Z_m y_m$
4. Restart.
   if satisfied, stop else $x_0 = x_m$, goto step 1.

The preconditioner is incorporated in step 2, where M^{-1} is an approximation of the matrix A^{-1}. In fact, M^{-1} is never calculated, because only the vector product $z_j = M^{-1}v_{j-1}$ is required. This can be derived by any approximate method. An important advantage of the FGMRES method is that M^{-1} can be the same as that arising from an implementation of ILU or MILU, but in addition (and more

importantly) it can be different at different iteration steps. In fact, the present FGMRES uses s-GMRES as its preconditioner. The equation $Az_j = v_{j-1}$ is solved using the s-GMRES method to give an approximate solution for z_j.

It is clear that in the FGMRES context the above-mentioned ILU method can be simply incorporated in a more general preconditioning approach, although more memory might be required than in the case of using solely the ILU technique. However, the FGMRES algorithm makes it very easy to switch between different preconditioners.

3.2.3. The GMRESR Preconditioner. Another effective preconditioning strategy was proposed by Van de Vorst and Vuik.[8] By investigating the low-rank transformation of A, GMRES is used as a preconditioner. A typical realization of this approach is the GMRES-GCR scheme, which is often referred to simply as GMRESR.

Algorithm.

1. Start $r_0 = b - Ax_0, k = -1$
2. Iteration
   for $k = 1, \ldots, m$ do
   - $u_k = M^{-1} r_k$
   - $c_k = Au_k$
   - For $i = 0, \ldots, k-1$,
     $\alpha = (c_k, c_i)$
     $c_k = c_k - \alpha c_i$
     $u_k = u_k - \alpha u_i$
   - Compute $c_k = c_k / \|c_k\|_2$
     $u_k = \dfrac{u_k}{\|c_k\|_2}$
   - Form the approximate solution;
     $x_{k+1} = x_k + u_k c_k^{\mathrm{H}} r_k$
     $r_{k+1} = r_k - c_k c_k^{\mathrm{H}} r_k$
3. Restart,
   if satisfied, stop else $x_0 = x_m$, goto step 1.

Here M^{-1} is the approximation of the matrix A^{-1}. The calculation is realized by the s-GMRES method, solving approximately $Mx = r_k$ and returning the vector u_k, just like the procedure used in the FGMRES scheme.

3.2.4. Convergence criterion. The convergence criterion used when solving the mild slope equation is based on the residue being defined as

$$Residue = \frac{(\|\nabla^2 \phi + k_c^2\|_2)^2}{(\|\nabla \phi^2\|_2)^2}. \tag{17}$$

The reason for using the squares of the 2-norms is to enable direct comparison with hitherto available solutions of the mild slope equation based on elliptic solvers.

3.3. The non-liner solver

Non-linear solvers using GMRES have been theoretically investigated by Brown and Saad.[18] The iteration is simply carried out according to the classical Newton method.

The non-linear algorithm is as follows.

1. Start $x = x_0$
2. Iteration $n = 0, 1, 2, \ldots$ until convergence
   - Solve $J(x_n)\delta x_n = -F(x_n)$
   - $x_{n+1} = x_n + \delta x_n$.

Here J is the system Jacobian and $F(x)$ represents the systems of equations. The solution process is carried out at two levels, the first corresponding to the solution of the system of linear equations and the second to the non-linear updating process. Therefore two separate convergence criteria must be defined: linear residue

$$Residue = \frac{(\|\nabla^2\psi + \nabla\psi\nabla\psi + k_c^2\|_2)^2}{(\|\nabla\psi^2\|_2)^2} \tag{18}$$

and non-linear residue

$$Residue - \frac{\|\psi_{new} - \psi_{old}\|_2}{\|\psi_{old}\|_2}. \tag{19}$$

4. TEST-CASES

Two typical cases are presented in this section. The first is Berkhoff's shoal. For this case both the linear and non-linear models are tested. The second test-case relates to wave reflection within a harbour, where only the linear model is tested since crossing waves exist.

4.1. Berkhoff's shoal

The previously discussed schemes are applied to Berkhoff's shoal, for which experimental test results are well documented. Results are compared with experimental data.

The lay-out of the shoal is shown in Figure 1. The shoal was built on a mild slope. The incident waves have a period of 1 s and are incident normally to the offshore boundary. The numerical model was run on a grid of 220×200 with 0·1 m grid spacing. The co-ordinate system used is such that the centre of the shoal is at $x = 10{\cdot}0$ m, $y = 11{\cdot}0$ m. The results from the linear model are shown in Figure 2 and those from the non-linear model are shown in Figure 3. Various transects are shown in Figures 4(a)–4(d) at $y = 12, 16, 20$ m and $x = 10$ m respectively, together with the experimental data as prescribed by Berkhoff *t al.*[19] The numerical results are in good agreement with the experimental data and the observed trend is in very close agreement with those from hithertoo available linear models.

The solution of the non-linear form of the governining equation produces similar results, but a much larger grid spacing can be used (0·4 m is used here). Results (plane plot, Figure 3, and the transects plotted together with the linear form results in Figure 4) are similar to those from the linear model as well as the experimental data, but some discrepancies appear possibly owing to three reasons: (i) the grid spacing is large, so the boundary conditions introduce larger errors; (ii) the use of the upwind dissipation scheme; (iii) the inaccuracy introduced by the non-linear formulation at wave-crossing points as pointed out by Radder.[12] It should be noted that this non-linear form of the equation is only a transformation of the linear Helmholtz equation and that it still describes linear waves. The present model results are in good agreement with the results from the multigrid model using the same non-linear form of the equation as presented by Li and Anastasiou.[9]

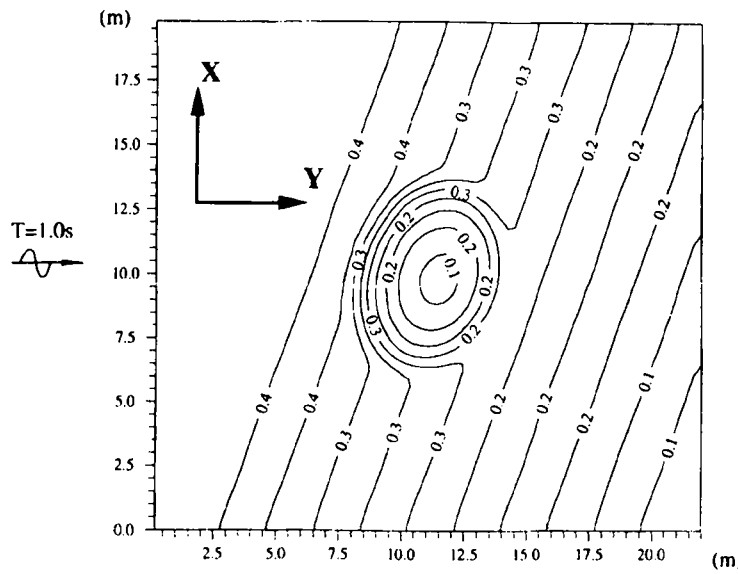

Figure 1. Geometry of Berkhoff's experiment

In order to clarify the results from the non-linear model, a further test was carried out. The grid spacing was reduced to 0·2 m. The results tend to approach the linear model results in the region near the shoal as can be seen in Figures 4(a), 4(b) and 4(d), but further downstream the results remain unchanged as shown in Figure 4(c). In this region the results are severely affected by the wave crossing, where the non-linear formulation becomes inaccurate as shown by Radder.[12]

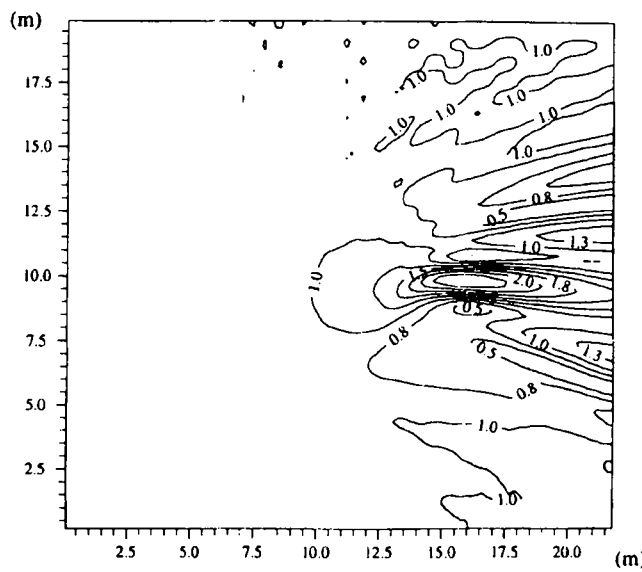

Figure 2. Contour plot of computed results of linear model

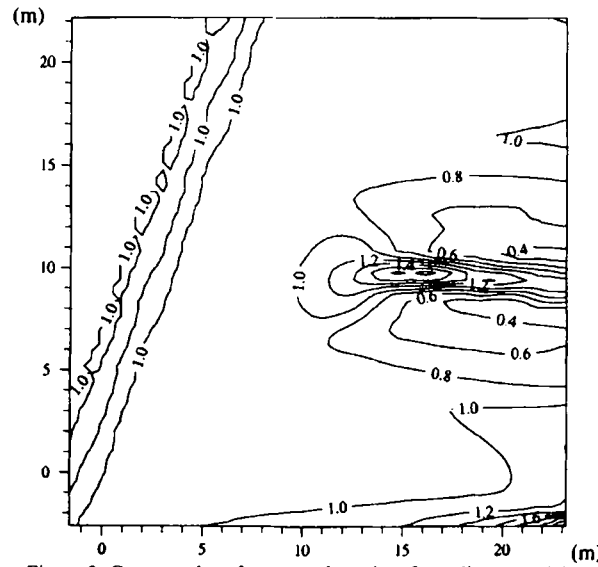

Figure 3. Contour plot of computed results of non-linear model

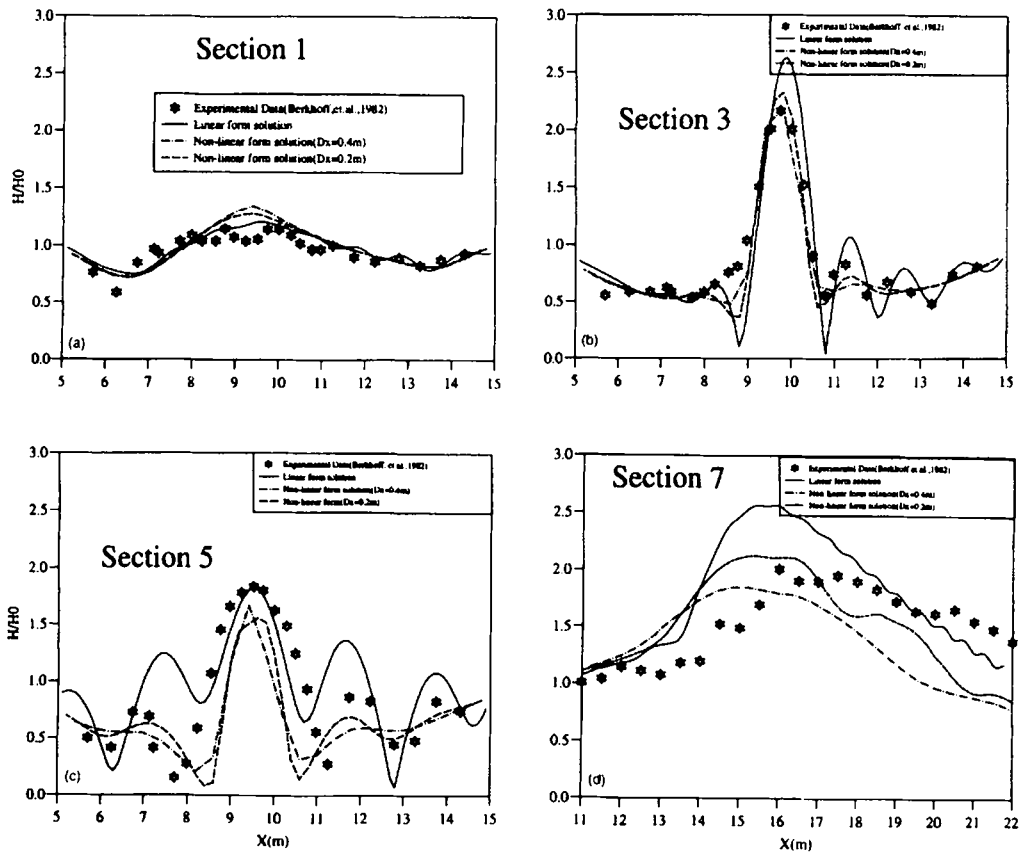

Figure 4. Comparison of relative wave height over four transects

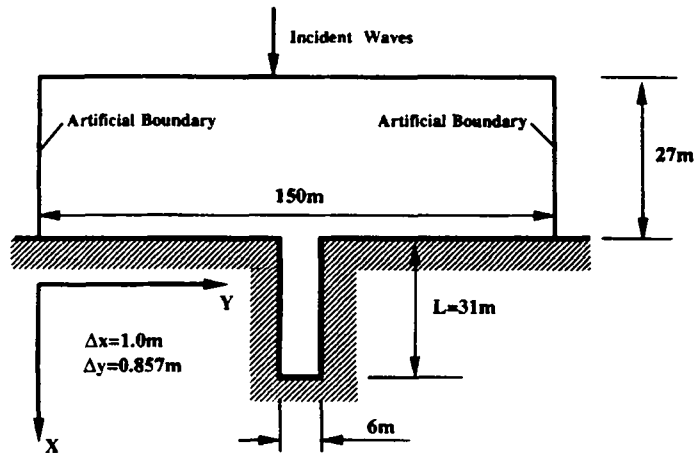

Figure 5. Lay-out for harbour resonance simulation

4.2. *Wave reflections within a harbour*

In order to test the reflection characteristics of the code, a harbour case for which theoretical results are available is tested. The grid lay-out is very similar to that of Panchang *et al.*[4] as shown in Figure 5. The resonance at the centre of the back wall of the harbour is shown in Figure 6, together with the theoretical results of Lee.[20] Close agreement is reached for two resonance peaks.

4.3. *Convergence*

The convergence curves in terms of multiplication counts for various schemes for Berkhoff's shoal are shown in Figure 7. The number of multiplications is used as the criterion, since either the total number of iterations or the matrix–vector product count cannot reveal the full extent of the

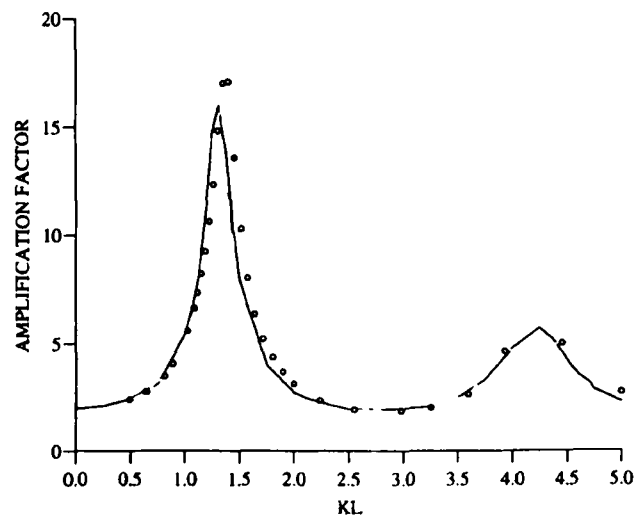

Figure 6. Wave height comparison for harbour resonance. —, Theoretical results;[20] o, modelling results

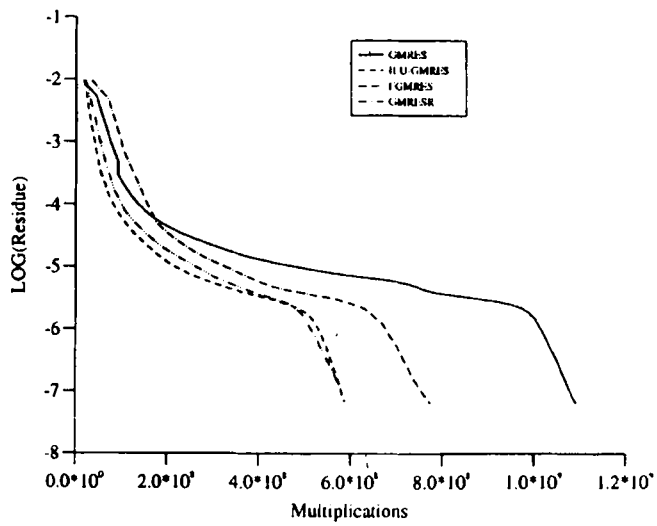

Figure 7. Comparisons of convergence of various schemes

computational load associated with each scheme. The Arnoldi iteration number m for the various schemes was adjusted to that they all had approximately equal computational memory requirements. The s-GMRES scheme used $m = 15$ and so did the ILU-GMRES scheme. The FGMRES scheme used $m = 10$ at the outer iterations and $m = 5$ at the inner iterations. CMRESR used $m = 6$ at the innter GMRES iterations and $m = 6$ at the outer GCR iterations.

It can be seen that plain s-GMRES give good convergence, but the preconditioner considerably accelerates the convergence rate. The ILU, FGMRES and GMRESR preconditioners perform in a very similar way. All present models perform favourably compared with other available published results. The preconditioned CG method applied to the normal equations as presented by Panchang *et al.*[4] used more than 2000 global iterations, with each iteration requiring approximately $27N$ (N is the total number of grid points) multiplications. Thus the total multiplication count for this case was $2{\cdot}376 \times 10^9$. The GCG method of Li[21] requires more than 2300 global iterations to reach the same convergence and within each iteration the multiplication count is $21N$. Thus the total multiplication count is $2{\cdot}125 \times 10^9$. It should be noted that GCG could be accelerated if equivalent memory to the present GMRES(m) were to be utilized which would enable GCG(m) to be used. However, it has been demonstrated by Saad and Schultz[6] that GMRES still outperforms GCR(m) in terms of efficiency.

In order to show the monotonic convergence of the present schemes. Berkhoff's shoal case was run down to machine accuracy (10^{-16}) for the linear model. The curve is shown in Figure 8. It can be seen that monotonic convergence is indeed achieved.

The convergence of the non-linear scheme is plotted in Figure 9, which shows the convergence of both the linear and non-linear residues. For these tests GMRES(m) with $m = 5$ was used.

In Figure 10 the iteration count of ILU-GMRES for the harbour resonance problem is given. For different frequencies the iteration number count can be very different and so can the calculation time. As expected, the twin peaks in the curve of iteration counts closely follow those of the resonance curve (Figure 6). For these tests the residue limit was set as 10^{-10}.

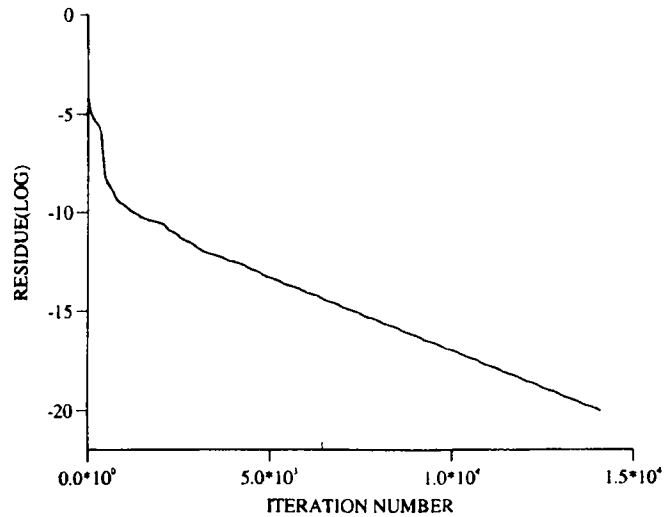

Figure 8. Convergence of ILU-GMRES

5. CONCLUSIONS

Models of iterative solutions of the mild slope equation implementing GMRES schemes have been developed and tested. The algorithm implementing the s-GMRES scheme shows good convergence. The s-GMRES method effectively avoids the upper Hessenberg factorization for the complex matrix arising from the present governing equation. Two well-known cases of nearshore wave propagation are tested and the results are favourable compared with available theoretical, numerical or experimental data. The rate of convergence of s-GMRES is further accelerated by various preconditioning strategies such as ILU, FGMRES and GMRESR. Results show that the preconditioned models, in particular ILU and FGMRES, perform much better than s-GMRES for the cases considered.

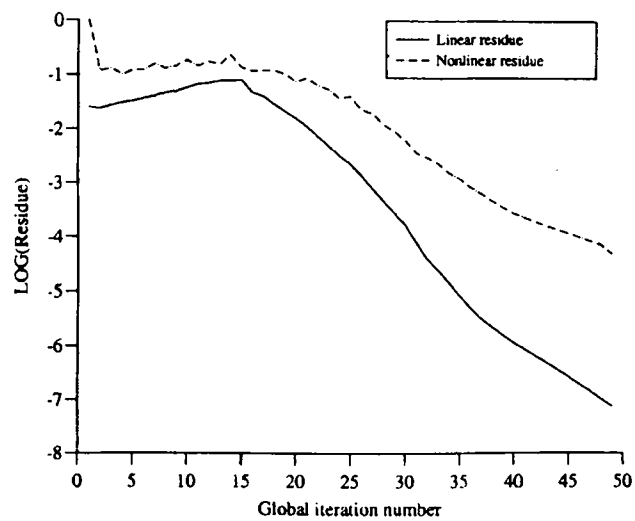

Figure 9. Convergence of non-lineaer GMRES

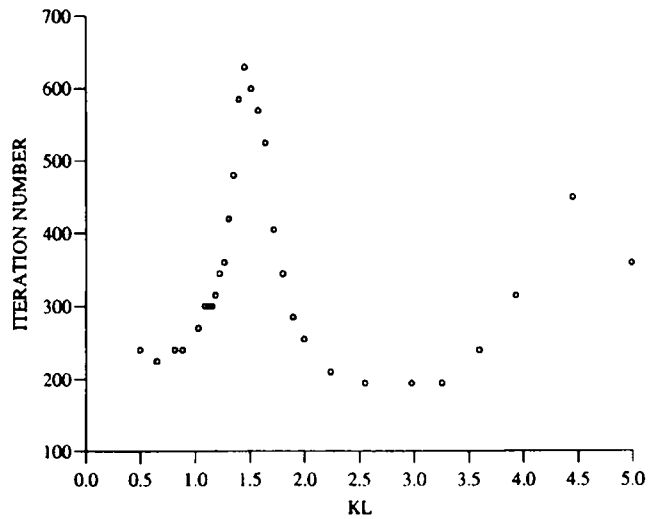

Figure 10. Performance of ILU-GMRES for harbour resonance calculations

APPENDIX

ET-p-GMRES	eigenvalue-transferred preconditioning GMRES[14]
FGMRES	flexible GMRES[7]
GCR	generalized conjugate gradient residue scheme[22]
GMRES	generalized minimum residue scheme[6]
GMRES(m)	m-step-truncated GMRES[6]
GMRESR	GMRES residue[8]
ILU	incomplete lower–upper factorization
s-GMRES	simpler GMRES[15]

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